MATH 1A - FINAL EXAM DELUXE - SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(10 points, 5 points each)* Find the following limits

(a) $\lim_{x\to\infty} \sqrt{x^2+1} - x$

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x^2+1} - x &= \lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} \\ &= \frac{1}{\infty} \\ &= 0\end{aligned}$$

(b) $\lim_{x\to 0^+} x^{x^2}$

1) Let $y = x^{x^2}$

2) Then $\ln(y) = x^2 \ln(x)$

3)

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}} \stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0$$

4) Hence

$$\lim_{x\to\infty} y = e^0 = 1$$

Date: Friday, August 12th, 2011.

2. *(10 points)* Use the **definition** of the derivative to calculate $f'(x)$, where:

$$f(x) = x^2$$

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{x^2 - a^2)}{x - a} \\ &= \lim_{x \to a} \frac{(x-a)(x+a)}{x - a} \\ &= \lim_{x \to a} x + a \\ &= 2a \end{aligned}$$

Hence, $\boxed{f'(x) = 2x}$

3. *(10 points, 5 points each)* Find the derivatives of the following functions

(a) y', where $x^y = y^x$

Hint: Take $\ln$s first!
Taking $\ln$s:

$$y \ln(x) = x \ln(y)$$

Differentiating and solving for y':

$$\begin{aligned} y' \ln(x) + \frac{y}{x} &= \ln(y) + \frac{xy'}{y} \\ y' \left(\ln(x) - \frac{x}{y} \right) &= \ln(y) - \frac{y}{x} \\ y' &= \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}} \end{aligned}$$

(b) y' at $(0, 0)$, where $\sin(y) = x^2 - y^2$

Differentiating:

$$\cos(y)y' = 2x - 2yy'$$

Now plug in $x = 0$ and $y = 0$:

$$(1)y' = 0 - 0$$

$$y' = 0$$

4. *(15 points)* Assume Peyam's happiness function is given by:

$$H = M^2 L + 2G$$

Where:

- M is the happiness due to holding office hours
- L is the happiness due to lecturing
- G is the happiness due to grading exams

Assume that at the end of the summer:

- Peyam's happiness due to holding office hours is 5 utils/week, and is decreasing by 2 utils/week
- Peyam's happiness due to lecturing is 10 utils/week, and is decreasing by 1 util/week
- Peyam's happiness due to grading exams, is 2 utils/week, and is decreasing by 1 utils/week.

Question: By how much is Peyam's happiness increasing/decreasing at the end of the summer?

1) No picture needed

2) WTF $\frac{dH}{dt}$

3) $H = M^2 L + 2G$

4) Differentiating, we get:

$$\frac{dH}{dt} = 2M\frac{dM}{dt}L + M^2\frac{dL}{dt} + 2\frac{dG}{dt}$$

5) Plug in $M = 5$, $\frac{dM}{dt} = -2$, $L = 10$, $\frac{dL}{dt} = -1$, $\frac{dG}{dt} = -1$:

$$\frac{dH}{dt} = 2(5)(-2)(10) + (5)^2(-1) + 2(-1)$$

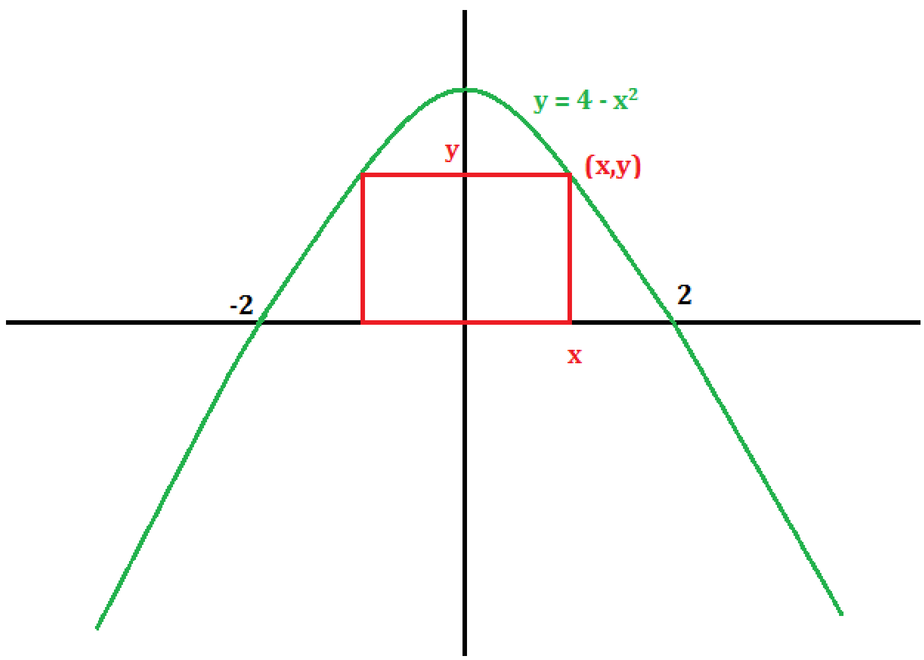
6)

$$\frac{dH}{dt} = -200 - 25 - 2 = -227$$

Peyam's happiness is decreasing by 227 utils/week. Fortunately, this doesn't correspond to reality :)

5. *(20 points)* What is the area of the largest rectangle that can be put inside the parabola $y = 4 - x^2$?

1) Picture:
1A/Math 1A Summer/Exams/FDrectangle.png



2) In the picture above, the length of the rectangle is $2x$ and the width is y, so the area is:

$$A = 2xy$$

Now (x, y) is on the parabola, so $y = 4 - x^2$, whence:

$$A(x) = 2x(4 - x^2) = 8x - 2x^3$$

3) Constraint: The constraint is $\boxed{0 \leq x \leq 2}$. (you find the 2 by solving $4 - x^2 = 0$)

4)

$$A'(x) = 8 - 6x^2 = 0 \Longleftrightarrow 6x^2 = 8 \Longleftrightarrow x^2 = \frac{8}{6} = \frac{4}{3} \Longleftrightarrow x = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

Now $A(0) = 0$ and $A(2) = 0$, so by the closed interval method, $A\left(\frac{2}{\sqrt{3}}\right)$ is the biggest area, and:

$$A\left(\frac{2}{\sqrt{3}}\right) = 2\left(\frac{2}{\sqrt{3}}\right)\left(4 - \left(\frac{2}{\sqrt{3}}\right)^2\right) = \frac{4}{\sqrt{3}}\left(4 - \frac{4}{3}\right) = \frac{4}{\sqrt{3}}\left(\frac{8}{3}\right) = \frac{32}{3\sqrt{3}}$$

6. *(15 points)*

(a) *(13 points)* Show that the following equation has exactly one solution:

$$\cos(x) = 2x$$

Let $f(x) = \cos(x) - 2x$

<u>At least one solution:</u> $f(0) = 1 - 0 = 1 > 0$, $f(\pi) = -1 - 2\pi < 0$, f is continuous, so by **the IVT**, f has at least one zero.

<u>At most one solution:</u> Suppose f has two zeros a and b. Then $f(a) = f(b) = 0$, so by **Rolle's theorem**, there is some c with $f'(c) = 0$.

But $0 = f'(c) = -\sin(c) - 2 < 0$, so $0 < 0$, contradiction!

Therefore, f has exactly one zero, and hence $\cos(x) = 2x$ has exactly one solution.

(b) *(2 points)* Use part (a) to show that the following function has exactly one critical point:

$$g(x) = \sin(x) - x^2$$

$g'(x) = \cos(x) - 2x = f(x)$. But we've shown in (a) that f has exactly one zero, hence $g'(x)$ has exactly one zero, that is g has exactly one critical point.

7. *(20 points)* Use the **definition** of the integral to evaluate:

$$\int_0^1 \left(x^3 - 2\right) dx$$

You may use the following formulas:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Note: -2 for not writing $\lim\limits_{n\to\infty}$

Preliminary work:

- $f(x) = x^3$
- $a = 0, b = 1, \Delta x = \frac{1-0}{n} = \frac{1}{n}$
- $x_i = \frac{i}{n}$

$$\begin{aligned}
\int_1^2 x^3 - 2dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right) \left(\left(\frac{i}{n}\right)^3\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n}\right) \left(\frac{i^3}{n^3}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} \\
&= \lim_{n\to\infty} \frac{1}{n^4} \left(\sum_{i=1}^{n} i^3\right) \\
&= \lim_{n\to\infty} \frac{1}{n^4} \left(\frac{n^2(n+1)^2}{4}\right) \\
&= \lim_{n\to\infty} \frac{(n+1)^2}{4} \\
&= \frac{1}{4}
\end{aligned}$$

Check: (not required, but useful)

$$\int_0^1 x^3 dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

8. *(30 points, 5 points each)* Find the following:

(a) $\int_{-1}^{1} \sqrt{1-x^2}dx$

Note: Don't spend too much time on this one, either you know it or you don't!

The integral represents the area of a semicircle of radius 1, hence:

$$\int_{-1}^{1} \sqrt{1-x^2}dx = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$$

(b) The antiderivative F of $f(x) = 3e^x + 4\sec^2(x)$ which satisfies $F(0) = 1$.
The MGAD of f is $F(x) = 3e^x + 4\tan(x) + C$. To find C, use the fact that $F(0) = 1$, so $3 + 0 + C = 1$, so $C = -2$, hence:

$$F(x) = 3e^x + 4\tan(x) - 2$$

(c) $g'(x)$, where $g(x) = \int_{x^2}^{e^x} \sin(t^3)dt$
Let $f(t) = \sin(t^3)$, then: $g(x) = F(e^x) - F(x^2)$, so:

$$g'(x) = F'(e^x)(e^x) - F'(x^2)(2x) = f(e^x)e^x - f(x^2)(2x) = \sin\left(e^{3x}\right)e^x - \sin\left(x^6\right)(2x)$$

(d) $\int (\cos(x))^3 \sin(x)dx$
Let $u = \cos(x)$, then $du = -\sin(x)dx$, so:

$$\int (\cos(x))^3 \sin(x)dx = \int u^3(-du) = -\frac{u^4}{4} + C = -\frac{(\cos(x))^4}{4} + C$$

(e) $\int_e^{e^2} \left(\frac{(\ln(x))^3}{x}\right) dx$

Let $u = \ln(x)$, then $du = \frac{1}{x}dx$, and $u(e) = \ln(e) = 1$ and $u(e^2) = 2$, so:

$$\int_e^{e^2} \left(\frac{(\ln(x))^3}{x}\right) dx = \int_1^2 u^3 du = \left[\frac{u^4}{4}\right]_1^2 = \frac{15}{4}$$

(f) The average value of $f(x) = \sin\left(x^5\right)\left(\cos\left(x^2\right) + e^{x^2} + x^4\right)$ on $[-\pi, \pi]$

$$\frac{\int_{-\pi}^{\pi} \sin\left(x^5\right)\left(\cos\left(x^2\right) + e^{x^2} + x^4\right) dx}{\pi} = \frac{0}{2\pi} = 0$$

because f is an odd function.

9. *(10 points)* Find the area of the region enclosed by the curves:

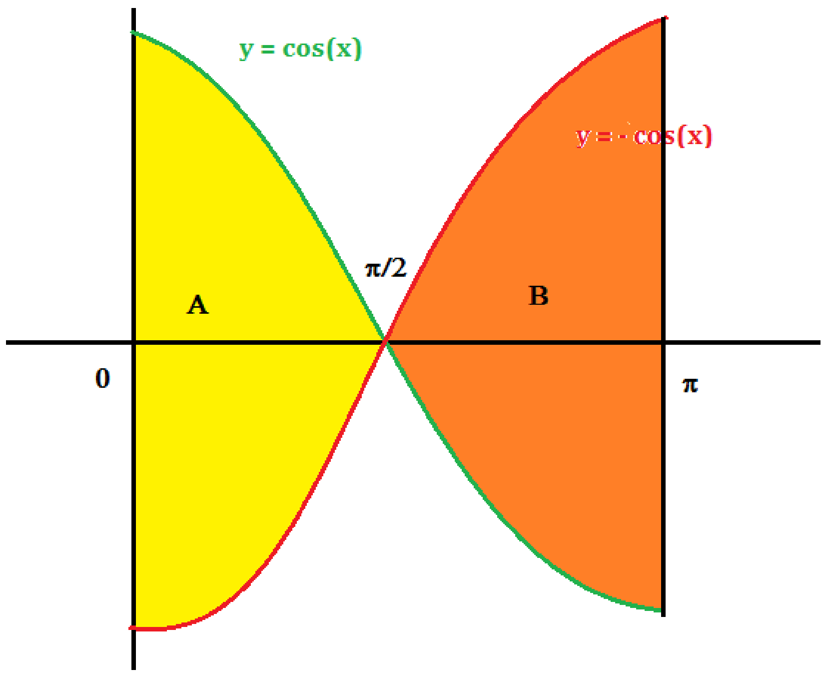
$$y = \cos(x) \quad \text{and} \quad y = -\cos(x)$$

from 0 to π.

Hint: It might help to notice a certain symmetry in your picture!

<u>Picture:</u>

1A/Math 1A Summer/Exams/Finalarea.png



Then determine the points of intersection between the two curves:

$$\begin{aligned} \cos(x) &= -\cos(x) \\ 2\cos(x) &= 0 \\ \cos(x) &= 0 \\ x &= \frac{\pi}{2} \end{aligned}$$

On $\left[0, \frac{\pi}{2}\right]$, $\cos(x)$ is above $-\cos(x)$, and on $\left[\frac{\pi}{2}, \pi\right]$, $-\cos(x)$ is above $\cos(x)$, so we'll have to figure out $A + B$ as in the picture. However, notice the symmetry! Namely, $A = B$, so all we really need to calculate is $A + B = 2A$, that is:

$$
\begin{aligned}
2\int_0^{\frac{\pi}{2}} \left(\cos(x) - (-\cos(x))\right) dx & \\
&= 2\int_0^{\frac{\pi}{2}} 2\cos(x)dx \\
&= 4\int_0^{\frac{\pi}{2}} \cos(x)dx \\
&= 4\left[\sin(x)\right]_0^{\frac{\pi}{2}} \\
&= 4(1-0) \\
&= 4
\end{aligned}
$$

10. *(10 points)* If $f(x) = \frac{x^3}{3} - \frac{x^2}{2}$, find:

(a) Intervals of increase and decrease, and local max/min

$f'(x) = x^2 - x = x(x-1) = 0$ if $x = 0, 1$.

Now drawing a sign table, you should see that:
f is $\boxed{\textbf{increasing on } (-\infty, 0)\textbf{, decreasing on } (0,1)\textbf{, and increasing on } (1,\infty)}$.

And hence f has a local max $\boxed{f(0) = 0}$, and a local min $\boxed{f(1) = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}}$

(b) Intervals of concavity and inflection points (just give me the $x-$coordinate of the IP)

$f''(x) = 2x - 1 = 0$ if $x = \frac{1}{2}$.

Hence f is $\boxed{\textbf{concave down on } \left(-\infty, \frac{1}{2}\right) \textbf{ and concave up on } \left(\frac{1}{2}, \infty\right)}$.
Moreover, f has an inflection point at $\boxed{x = \frac{1}{2}}$.

Bonus 1 *(5 points)* Fill in the gaps in the following proof that the function f is not integrable on $[0,1]$:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Step 1: Pick x_i^* such that x_i^* **is rational**. Then:

$$\begin{aligned} \int_0^1 f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^n \Delta x f(x_i^*) \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}(0) \\ &= \lim_{n\to\infty} 0 \\ &= 0 \end{aligned}$$

Step 2: Pick x_i^* such that x_i^* **is irrational**. Then:

$$\begin{aligned} \int_0^1 f(x)dx &= \lim_{n\to\infty} \sum_{i=1}^n \Delta x f(x_i^*) \\ &= \lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n}(1) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n 1 \\ &= \lim_{n\to\infty} \frac{1}{n}(n) \\ &= \lim_{n\to\infty} 1 \\ &= 1 \end{aligned}$$

Since we get two different answers for the integral, we have a contradiction. $\Rightarrow\Leftarrow$. And hence f is not integrable on $[0,1]$.

Note: See the handout 'Integration sucks!!!' for a nice discussion of this problem!

Bonus 2 *(5 points)* Another way to define $\ln(x)$ is:

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

Show **using this definition only** that $\ln(e^x) = x$.

Hint: Let $g(x) = \ln(e^x) = \int_1^{e^x} \frac{1}{t} dt$. Differentiate g, simplify, and antidifferentiate. Make sure you face the issue of the constant!

Let $f(t) = \frac{1}{t}$, then $g(x) = F(e^x) - F(1)$, so:

$$g'(x) = F'(e^x)(e^x) - 0 = f(e^x)(e^x) = \frac{1}{e^x} e^x = 1$$

Since $g'(x) = 1$, we get $g(x) = x + C$. To figure out what C is, plug in $x = 0$, and we get:

$$\begin{aligned} g(0) &= 0 + C \\ \int_1^{e^0} \frac{1}{t} dt &= C \\ \int_1^1 \frac{1}{t} dt &= C \\ 0 &= C \\ C &= 0 \end{aligned}$$

Hence $g(x) = x$, so $\boxed{\ln(e^x) = x}$

Bonus 3 *(5 points)* Define the **Product integral** $\prod_a^b f(x)dx$ as follows:

If we define Δx, x_i, and x_i^* as usual, then:

$$\prod_a^b f(x)dx = \lim_{n\to\infty} \left(f(x_1^*)\right)^{\Delta x} \left(f(x_2^*)\right)^{\Delta x} \cdots \left(f(x_n^*)\right)^{\Delta x}$$

(that is, instead of summing up the $f(x_i^*)$, we just multiply them!)

Show that this is nothing new, that is, express $\prod_a^b f(x)dx$ in terms of $\int_a^b f(x)dx$

Hint: How do you turn a product into a sum?

Let $P = \prod_a^b f(x)dx$. Then:

$$\begin{aligned}
\ln(P) &= \ln\left(\lim_{n\to\infty} \left(f(x_1^*)\right)^{\Delta x} \left(f(x_2^*)\right)^{\Delta x} \cdots \left(f(x_n^*)\right)^{\Delta x}\right) \\
&= \lim_{n\to\infty} \ln\left(\left(f(x_1^*)\right)^{\Delta x} \left(f(x_2^*)\right)^{\Delta x} \cdots \left(f(x_n^*)\right)^{\Delta x}\right) \\
&= \lim_{n\to\infty} \ln\left(f(x_1^*)^{\Delta x}\right) + \ln\left(f(x_2^*)^{\Delta x}\right) + \cdots + \ln\left(f(x_n^*)^{\Delta x}\right) \\
&= \lim_{n\to\infty} \Delta x \ln\left(f(x_1^*)\right) + \Delta x \ln\left(f(x_2^*)\right) + \cdots + \Delta x \ln\left(f(x_n^*)\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \ln\left(f(x_i^*)\right) \\
&= \int_a^b \ln(f(x))dx
\end{aligned}$$

So $\ln(P) = \int_a^b \ln(f(x))dx$, so $P = e^{\int_a^b \ln(f(x))dx}$, hence:

$$\prod_a^b f(x)dx = e^{\int_a^b \ln(f(x))dx}$$